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Results on stabilization of nonlinear systems under finite data-rate constraints

Claudio De Persis

Abstract—We discuss in this paper a result concerning the stabilization problem of nonlinear systems under data-rate constraints using *output* feedback. To put the result in a broader context, we shall first review a number of recent contributions on the stabilization problem under data-rate constraints when full-state measurements are available.

I. INTRODUCTION

By data-rate constraints it is meant that the analog measurements taken at the system are sampled and converted into packets comprising a finite number of bits. Hence, the feedback signal which drives the controller is sampled and quantized. In the paper, we shall refer to this kind of feedback signal as *encoded* ([1]) feedback. With respect to the case of linear systems, examined in a number of papers (to cite a few, [1], [2], [3] [4], [5], [6], [7], [8], and references therein), the nonlinear stabilization problem under data-rate constraints is more subtle because of the well-known difficulties to have robustness with respect to quantization (measurement) errors, and because the characterization of the behavior of the nonlinear system during the inter-sampling period, when no new information is provided, is harder. This is especially true when dealing with nonlinear *continuous-time* systems in the presence of arbitrary sampling time. Among very recent contributions devoted to the nonlinear stabilization problem under data-rate constraints, we are particularly interested in the paper [9], where the problem is studied under a robustness property enjoyed by the system. In particular the authors assume input-to-state stability with respect to measurement errors. In this paper, we review the efforts in [10] and [11] to relax the assumptions in [9]. In particular, following [10], we first consider nonlinear systems which are integral input-to-state stabilizable, a property which is known to be less restrictive than input-to-state stability – see e.g. [12]. Then, following [11], we consider nonlinear systems which are asymptotically stabilizable. In both cases we adopt *dynamic* quantizers which allow to achieve *asymptotic* (and not only practical) stabilizability by encoded feedback and we also provide conditions on the data-rate under which the results can be achieved. Note that we do not deal with the minimality of the data rate. This issue has been elegantly

investigated in [13] using topological arguments. See [14] and [15] for other approaches to the problem. Finally, we introduce a new class of encoders which allow us to achieve asymptotic stabilization under data-rate constraints when only output measurements are available.

Generalities about the problem of stabilization by encoded feedback are presented in Section II. Solutions to the problem under integral input-to-state and asymptotic stabilizability, are reviewed in Section III. These solutions assume that full-state measurements are available. The case of partial-state measurements is studied in Section IV for observable nonlinear systems. Conclusions are drawn in the last section.

II. PRELIMINARIES

Consider a system of the form

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x)\end{aligned}\quad (1)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $f(\cdot, \cdot)$, $h(\cdot)$ smooth maps. We shall assume that the compact set to which the initial condition $x(0)$ belongs is a known hyper-cube in \mathbb{R}^n , denoted C_0^X , with centroid the origin $0 \in \mathbb{R}^n$ and edges of length $2X$, where X is a non-negative real number. In particular, we introduce: ¹

Assumption 1: A positive constant X is known such that: $|x(0)|_\infty \leq X$. \triangleleft

We shall denote by $W \geq X$, $\bar{W} := W + X$, F , U some positive real numbers for which

$$|f(x, u) - f(\bar{x}, u)|_\infty \leq F|x - \bar{x}|_\infty \quad (2)$$

for all x, \bar{x}, u such that

$$|x|_\infty, |\bar{x}|_\infty \leq \bar{W}, \quad |u|_\infty \leq U. \quad (3)$$

We shall examine in the next section and in Section IV nonlinear systems (1) in three different scenarios. For each scenario we shall specify three different sets of values for W , \bar{W} , U and consequently F .

Having introduced the system, we proceed to introduce the so-called *quantization region* Ω , a subset of the state space \mathbb{R}^n on which the functioning of encoders and decoders is based on.

Quantization region. The quantization region Ω is a hyper-cube centered around the centroid \bar{x} in which each edge has length L — the latter is called the *range* of the

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¹For a vector $y \in \mathbb{R}^n$ the symbol $|y|_\infty$ denotes the ∞ -norm of vector y , i.e. the quantity $\max_{1 \leq i \leq n} |y_i|$.

quantization region. Hence, $\Omega = C_{\bar{x}}^{L/2}$. State x belongs to Ω if and only if $|x - \bar{x}|_{\infty} \leq L/2$. Both \bar{x} and L are updated every T units of time. The way in which this update takes place is established by the encoder and the decoder — see formulas (8), (9) below. Each edge of the quantization region is uniformly partitioned into the same number $N > 1$ of parts. Therefore the quantization region will be uniformly partitioned into N^n sub-regions or smaller hyper-cubes, each one with its own centroid easily computable from the knowledge of \bar{x} , L and N . The number N is used to define the quantity $R := \Lambda/N$, with $\Lambda := e^{FT}$.

Remark. Observe that number R can be set equal to any (arbitrarily small) positive number by increasing the number of bits used to encode the state information. Indeed, this number of bits is given by $B := \lceil \log_2(N^n) \rceil$. Having n — the dimension of the state space of the system — fixed, increasing B means increasing N . \triangleleft

The encoder/decoder update equations for the centroid and the range will be given in the following sections for each scenario we consider. In the meanwhile we describe in more detail the qualitative functioning of the encoder and the decoder.

Encoder. At each time kT , quantities $x(kT)$, $\bar{x}(kT^-) := \lim_{t \rightarrow kT^-} \bar{x}(t)$ and $L(kT)$ are available to the encoder which uses them to construct the quantization region $\Omega(kT)$. If $x(kT) \in \Omega(kT)$, the encoder determines the sub-region of the quantization region where $x(kT)$ lies. This sub-region has a centroid $\hat{x}(kT)$ whose expression is given by

$$\hat{x}(kT) = \bar{x}(kT^-) + \begin{bmatrix} \tilde{b}_1(kT)L(kT)/2N \\ \vdots \\ \tilde{b}_n(kT)L(kT)/2N \end{bmatrix} \quad (4)$$

where the $\tilde{b}_i(kT)$'s are suitable integers taking values in the set

$$\{-(N-1), \dots, -5, -3, -1, +1, +3, +5, \dots, +(N-1)\}$$

if N is an even integer, or in the set

$$\{-(N-1), \dots, -6, -4, -2, 0, +2, +4, +6, \dots, +(N-1)\}$$

if N is an odd integer. In either case, vector

$$\tilde{b}(kT) := [\tilde{b}_1(kT) \ \tilde{b}_2(kT) \ \dots \ \tilde{b}_n(kT)] \in \mathbb{R}^n$$

can take on N^n possible values. If $x(kT) \notin \Omega(kT)$, that is $x(kT)$ lies in the overflow region $\mathbb{R}^n \setminus \Omega(kT)$, then $\tilde{b}(kT)$ must take on an additional value denoting overflow. Hence, $\tilde{b}(kT)$ can be represented by a binary number if $\lceil \log_2(N^n + 1) \rceil$ bits are used. The binary number representing $\tilde{b}(kT)$ is indeed the symbol $s(kT)$ to be sent through the channel. We do not proceed further to specify the actions taken by the encoder and the decoder in the event that an overflow occurs because, it can be seen that (see e.g. [11], [10], [16]), by construction, overflow is guaranteed to never occur.

Decoder. The decoder at the other end of the channel performs an inverse operation with respect to the one performed by the encoder. If the received symbol $s(kT)$ denotes overflow, then the decoder infers that overflow is occurring. Otherwise, the decoder reconstructs the vector $\hat{x}(kT)$ from $s(kT)$. First of all, from $s(kT)$ the vector $\tilde{b}(kT)$ can be promptly derived and therefore $\hat{x}(kT)$ can be calculated by (4) once $\bar{x}(kT^-)$ becomes available. Vector $\bar{x}(kT^-)$ is indeed available to the decoder, for it implements the same update laws (8), (9) as the encoder.

Controller. The controller candidate to solve the stabilization problem by encoded state feedback is “inspired” by the principle of certainty equivalence. This means that, denoted by $\kappa(\cdot)$ the map which defines the stabilizing feedback control law, the encoded feedback control law can be chosen as

$$u = \kappa(\xi), \quad (5)$$

where ξ is the feedback signal generated by the encoder and the decoder (in Section III, $\xi = \bar{x}$, whereas in Section IV, $\xi = \varsigma^*$, with both \bar{x} and ς^* to be defined below). The remaining sections of the paper will be concerned with showing that controller (5) asymptotically stabilizes system (1) under three different scenarios.

III. STABILIZATION BY ENCODED FEEDBACK UNDER ISS AND GAS ASSUMPTIONS USING FULL-STATE MEASUREMENTS

In this section we consider the stabilization problem for systems (1) which can be made integral input-to-state stable (iISS – cf. [12]) and, respectively, globally asymptotically stable by state feedback. The reason to deal with the two classes of systems in parallel is that they share many common features among which the same equations for the encoder (and the decoder).

Let us now introduce the following:

Assumption 2: Full-state measurements are available, i.e. $h(x) = x$ in (1). \triangleleft

Assumption 3: There exist a smooth map $\kappa(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$, class- \mathcal{K}_{∞} functions $\varpi(\cdot)$, $\gamma(\cdot)$ and a class- \mathcal{KL} function $\beta(\cdot)$ for which the response of the closed-loop system

$$\dot{x} = f(x, \kappa(x + e)), \quad (6)$$

satisfies the inequality:

$$\varpi(|x(t)|_{\infty}) \leq \beta(|x(0)|_{\infty}, t) + \int_0^t \gamma(|e(\tau)|_{\infty}) d\tau,$$

for all $t \geq 0$. It is additionally required ([17]) for $\gamma(\cdot)$ to be a class- \mathcal{H}_I function, that is

$$\int_0^1 \frac{\gamma(s)}{s} ds < \infty. \quad \triangleleft$$

Assumption 4: There exist a smooth feedback law $u = \kappa(x)$, a smooth function $V(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and class- \mathcal{K}_{∞}

functions $\underline{\alpha}(\cdot)$, $\bar{\alpha}(\cdot)$ and $\alpha(\cdot)$ for which

$$\begin{aligned} \underline{\alpha}(|x|_\infty) &\leq V(x) \leq \bar{\alpha}(|x|_\infty) \\ \frac{\partial V}{\partial x} f(x, \kappa(x)) &\leq -\alpha(|x|_\infty). \end{aligned} \quad (7)$$

Remark. Clearly Assumption 4 is weaker than Assumption 3. Analogously, Assumption 3 is weaker than the ISS assumption in [9]. \triangleleft

The encoder/decoder update equations for the centroid and the range are now introduced. The centroid update law is:

$$\begin{aligned} \frac{d}{dt} \bar{x}(t) &= f(\bar{x}(t), u(t)), \quad t \in [kT, (k+1)T], \quad k \geq 0, \\ \bar{x}(kT) &= \hat{x}(kT), \quad k \geq 0, \end{aligned} \quad (8)$$

with initial condition $\bar{x}(0^-) = 0$ and where $\hat{x}(kT)$ is the centroid of the sub-region of $\Omega(kT)$ — defined by the centroid $\bar{x}(kT^-)$ and range $L(kT)$ — where $x(kT)$ lies. (See below for more details on how $\hat{x}(kT)$ is determined.) The range update law is:

$$\begin{aligned} L((k+1)T) &= RL(kT), \quad k \geq 0, \\ L(0) &= 2X. \end{aligned} \quad (9)$$

Depending on which one of the two assumptions above holds we have different values for the constants W, \bar{W}, U and therefore different values for the constant R in (9). In particular, if Assumption 3 holds, then define:

$$\begin{aligned} W &> \varpi^{-1} \left(\beta(X, 0) + \int_0^X \frac{\gamma(s)}{s} ds \right), \quad \bar{W} := W + X, \\ U &:= \max_{x: |x|_\infty \leq \bar{W}} |\kappa(x)|_\infty + 1. \end{aligned}$$

On the other hand, if Assumption 4 holds, set

$$c := \bar{\alpha}(X), \quad (10)$$

and introduce:

$$\begin{aligned} W &:= \underline{\alpha}^{-1}(c+1), \quad \bar{W} := W + X, \\ U &:= \max_{x: |x|_\infty \leq \bar{W}} |\kappa(x)|_\infty + 1. \end{aligned} \quad (11)$$

The stabilization result under Assumption 3 can be stated as follows:

Proposition 1: Let Assumptions 1, 2 and 3 hold. If

$$N > \Lambda e^T = e^{(F+1)T}, \quad (12)$$

then the solution of the closed-loop system

$$\dot{x} = f(x, \kappa(\bar{x}))$$

from the initial condition $|x(0)|_\infty \leq X$ and with $\bar{x}(\cdot)$ generated by the encoder/decoder (8) (with $u(\cdot) = \kappa(\bar{x}(\cdot))$), (9) satisfies:

- (i) For each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $|x(0)|_\infty \leq \bar{L}/2 \leq \delta(\varepsilon)$ implies $|x(t)|_\infty \leq \varepsilon$ for all $t \geq 0$;
- (ii) For each $\varepsilon > 0$, there exists $t(\varepsilon) > 0$ such that $|x(t)|_\infty \leq \varepsilon$ for all $t \geq t(\varepsilon)$.

Proof: See [10]. \blacksquare

By exploiting a result ([11] – see also [9]) on the exponential convergence of the encoded estimate $\bar{x}(\cdot)$ to the state $x(\cdot)$ whenever the latter evolves over a compact set, it is shown that the state $x(\cdot)$ cannot leave the set

$$\{x \in \mathbb{R}^n : |x|_\infty \leq W\}$$

because of the choice of N and the iISS property holding with a class- \mathcal{H}_I function $\gamma(\cdot)$.

In order to state the stabilization result under Assumption 4, additional notation is required. Rewrite the closed-loop system (1), (5) as follows:

$$\begin{aligned} \dot{x} &= f(x, \kappa(x)) + f(x, \kappa(\bar{x})) - f(x, \kappa(x)) \\ &= f(x, \kappa(x)) + g(x, \bar{x})(x - \bar{x}), \end{aligned} \quad (13)$$

where $g(\cdot, \cdot)$ is a suitable smooth map obtained from a straightforward application of the mean value theorem, and set:

$$M := \max_{\substack{x \in C_0^W \\ \chi \in C_0^{\bar{W}}}} \left| \frac{\partial V}{\partial x} g(x, \chi) \right|_\infty. \quad (14)$$

It is now possible to state:

Proposition 2: Let Assumptions 1, 2 and 4 hold. For each $\rho < c + 1$, if

$$N > \Lambda \text{ and } N \geq \Lambda(MX/\alpha \circ \bar{\alpha}^{-1}(\rho))^{2MXT}, \quad (15)$$

with M defined as in (14), then the solution of the closed-loop system (13)

$$\dot{x} = f(x, \kappa(\bar{x}))$$

from the initial condition $|x(0)|_\infty \leq X$ and with $\bar{x}(\cdot)$ generated by the encoder/decoder (8) (with $u(\cdot) = \kappa(\bar{x}(\cdot))$), (9) satisfies:

- (i) For each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $|x(0)|_\infty \leq L(0)/2 \leq \delta(\varepsilon)$ implies $|x(t)|_\infty \leq \varepsilon$ for all $t \geq 0$;
- (ii) For each $\varepsilon > 0$, there exists $t(\varepsilon) > 0$ such that $|x(t)|_\infty \leq \varepsilon$ for all $t \geq t(\varepsilon)$.

Proof: See [11]. \blacksquare

With respect to Proposition 1, the proof of the result above is based on the Lyapunov function introduced in Assumption 4. In particular a level set is chosen which contains the set of initial conditions of the system (1) and then the parameter N is fixed so as to make the time derivative of the Lyapunov function computed along the solution of the closed-loop system (13) strictly negative when the latter approaches the boundary of the level set. Before concluding this section, we note that, under the iISS assumption, a lower number of bits than under the stabilizability assumption may be employed for encoding when

$$\ln(MX/\alpha \circ \bar{\alpha}^{-1}(\rho)) > (2MXT)^{-1}.$$

Furthermore, the result in Proposition 1 can be applied more straightforwardly than the result in Proposition 2.

IV. ON STABILIZATION BY ENCODED FEEDBACK USING PARTIAL-STATE MEASUREMENTS

In this section we discuss the case in which Assumption 2 does not hold any longer and only partial-state measurements $y = h(x)$ are available.

A. Nonlinear observability

Being interested in solving a stabilization problem based on output measurements only, we recall a suitable notion of observability for nonlinear systems, namely the *uniform infinitesimal* observability property of [18], [19]:

Assumption 5: Maps $f(\cdot, \cdot)$ and $h(\cdot)$ are analytic and there exists a globally defined change of coordinates $\Phi(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which transforms system (1) into a system of the form:

$$\begin{aligned} \dot{x}_1 &= f_1(\underline{x}_1, x_2, u) \\ \dot{x}_2 &= f_2(\underline{x}_2, x_3, u) \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(\underline{x}_{n-1}, x_n, u) \\ \dot{x}_n &= f_n(x, u) \\ y &= f_0(x_1), \end{aligned} \quad (16)$$

where $\underline{x}_i := (x_1, \dots, x_i)$, and

$$\frac{\partial f_0}{\partial x_1} \neq 0, \quad \frac{\partial f_1}{\partial x_2} \neq 0, \quad \dots, \quad \frac{\partial f_{n-1}}{\partial x_n} \neq 0,$$

for all x_1, \dots, x_n, u .

Henceforth, for the sake of simplicity, we shall consider the case in which system (1) is already in the form (16). Although this section is concerned with the case in which the full state of system (1) can not be measured, we shall still let Assumption 1 hold.

As it is pointed out in [18], so far as the evolution of the vector $(x(t), u(t))$ remains in a compact set of the form

$$\mathcal{C} := \{(x, u) : |x|_\infty \leq W \text{ and } |u|_\infty \leq U\},$$

for some $W > 0, U > 0$, the properties:

- (i) Each function $f_i(\underline{x}_i, x_{i+1}, u)$ is globally Lipschitz (with Lipschitz constant G/\sqrt{n}) with respect to \underline{x}_i , uniformly in x_{i+1}, u ;
- (ii) There exist real numbers $0 < \alpha < \beta$ for which

$$\alpha \leq \left| \frac{\partial f_0}{\partial x_1} \right|, \quad \left| \frac{\partial f_1}{\partial x_2} \right|, \quad \dots, \quad \left| \frac{\partial f_{n-1}}{\partial x_n} \right| \leq \beta;$$

can be assumed to hold without loss of generality. In fact ([18]), for any choice of W and U , there exist functions $\varphi_0, \varphi_1, \dots, \varphi_n$ which agree over \mathcal{C} with f_0, f_1, \dots, f_n , respectively, and satisfy properties (i) and (ii). Then one could replace f_0, f_1, \dots, f_n with $\varphi_0, \varphi_1, \dots, \varphi_n$ and proceed with the analysis. Here, for the sake of simplicity, we assume that functions f_0, f_1, \dots, f_n already satisfy properties (i) and (ii) above over \mathcal{C} .

These properties are instrumental to design an observer for system (16). In fact, consider the pair of matrices

$$A(t) = \begin{pmatrix} 0 & a_2(t) & 0 & \cdots & 0 \\ 0 & 0 & a_3(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n(t) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad C(t) = \begin{pmatrix} a_1(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T$$

with

$$\alpha \leq a_i(t) \leq \beta, \quad i = 1, \dots, n.$$

Then, by Lemma 4.0 in [19], there exist a constant n -dimensional vector K and a symmetric positive definite matrix S satisfying

$$(A(t) - KC(t))^T S + S(A(t) - KC(t)) \leq -I.$$

Remark. In what follows we shall assume without loss of generality that, denoted by $\lambda_{\max}(S)$ the largest eigenvalue of S , $2\lambda_{\max}(S)G \geq 1$. \triangleleft

Let us now introduce the following:

Assumption 6: For any $X > 0$ and $U > 0$, there exists $W > 0$ such that $|x(0)|_\infty \leq X$ and $|u(t)|_\infty \leq U$ for all $t \geq 0$ implies $|x(t)|_\infty \leq W$ for all $t \geq 0$.

Under this assumption, Gauthier and Kupca have shown in [19] and [18] that the system:

$$\frac{d}{dt}\sigma(t) = f(\sigma(t), u(t)) + K_\theta(y - h(\sigma(t))), \quad |\sigma(t_0)|_\infty \leq X \quad (17)$$

where

$$K_\theta = \begin{bmatrix} \theta & 0 & \cdots & 0 \\ 0 & \theta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta^n \end{bmatrix} K,$$

and $\theta > 1$ is a constant gain, is an exponential observer for system (16). In particular, the following can be proven ([19], [18]):

Theorem 1: Let Assumptions 5 and 6 hold. Then, for any $\gamma > 0$, the observer gain $\theta = \lambda_{\max}(S)(\gamma + 2G) > 1$ is such that, for all $(x(0), \sigma(0))$ for which $|x(0)|_\infty, |\sigma(0)|_\infty \leq X$ and for all $u(\cdot)$ such that $|u(t)|_\infty \leq U$ for all $t \geq 0$, the response of system (16), (17) satisfies the inequality

$$|x(t) - \sigma(t)|_\infty \leq \mu(\gamma)e^{-\gamma t}|x(0) - \sigma(0)|_\infty \quad (18)$$

for all $t \geq 0$, where

$$\mu(\gamma) := \sqrt{n \frac{\lambda_{\max}(S)}{\lambda_{\min}(S)}} (2\lambda_{\max}(S)(\gamma + G))^{n-1},$$

with $\lambda_{\min}(S)$ the minimal eigenvalue of S , is a polynomial function in γ .

Having introduced the observability property, and keeping in mind Assumption 4, we take the constants W, U in this section equal to:

$$W := \underline{\alpha}^{-1}(c + 1), \quad U := \max_{|x|_\infty \leq W} |\kappa(x)|_\infty.$$

If we consider the evolution of the closed-loop system

$$\dot{x} = f(x, \kappa(x))$$

under measurement disturbance e , that is

$$\dot{x} = f(x, \kappa(x + e)) , \quad (19)$$

conveniently rewritten as

$$\dot{x} = f(x, \kappa(x)) + g(x, x + e)e , \quad (20)$$

with $g(\cdot, \cdot)$ as in Section III, then it is possible to assess an interval of time over which the solution is guaranteed to stay within a prescribed level set, provided that the variable $x + e$ is small enough. In fact, a straightforward Lyapunov-based argument shows the following result:

Lemma 1: Under Assumptions 1 and 4, there exists a time $\tau := (2MW)^{-1}$, with M as in (14), such that the response of system (19) satisfies $x(t) \in \Gamma_{c+1} := \{x : V(x) \leq c + 1\}$, and hence $|x(t)|_\infty \leq W$, for all $t \in [0, \tau]$, provided that $|x(t) + e(t)|_\infty \leq W$ for all $t \in [0, \tau]$.

Proof: The proof is straightforward and is omitted. It can be found e.g. in [20], [11]. ■

In the scenario in which state information is reconstructed starting from output measurements, which are encoded and transmitted from *time to time*, the controller must be able to provide an effective control action over the time interval² $[t_k, t_{k+1}]$ based on the feedback information provided at time t_k . However, the encoded information at time $t_0 = 0$ can be very poor regarding the initial state of the plant – as at t_0 the observer does not have enough time to reconstruct the internal state of the system – and so can be the control capability over the interval $[t_0, t_1]$. Reliable information concerning the state of the system becomes available with the arrival of the second packet of bits at time t_1 . This observation points out that time t_1 cannot be too far away from the origin if one is interested in maintaining the system under control. How early the second packet of bits must arrive is quantified by the following hypothesis:

Assumption 7: $t_1 \leq t_0 + \tau$. ◁

No special hypotheses are required on the arrival times of the successive packets. Thus, given any $T > 0$, we shall consider the case in which $t_{k+1} - t_k \geq T$ for each $k = 1, 2, \dots$

In what follows, we shall assume that a specific decay rate γ^* and the corresponding gain θ^* have been fixed in such a way that the exponential observer exists, i.e. inequality (18) holds with γ replaced by γ^* . In particular and without loss of generality, noting that t_1 depends on τ only, the latter being independent of γ and θ , and keeping in mind that $\mu(\gamma)$ is a polynomial function of γ , we can assume the

²In the previous sections we have considered uniformly spaced sampling intervals kT , with $k \geq 0$. Here, for reasons which will be clear in a moment, it is more convenient – especially for times close to the origin – to consider sampling intervals which may not always have the same length. Consequently, in this section we shall denote the sampling times by t_k , with $k \geq 0$.

decay rate γ^* , and hence θ^* , to have been fixed in such a way that

$$\mu(\gamma)e^{-\gamma t_1} \leq \mu(\gamma^*)e^{-\gamma^* t_1} , \text{ for all } \gamma \geq \gamma^* .$$

B. Encoding using nonlinear observers

We describe here the method to convert at each time step the analog information concerning the process (16) into packets of B bits in the present scenario of partial-state measurements. In fact, as no direct encoding of the internal state of (16) is possible, we take the approach suggested in [5], for linear discrete-time systems, to encode the *estimated* state of the system. To the purpose of encoding the estimated state of system (16), a reliable estimate must be available and the asymptotic observer recalled above can actually provide such an estimate – if suitable bounds on the state and the control input of (16) are fulfilled. Hence, we shall consider the case in which the asymptotic observer (17) is *embedded* in the encoder. In order to encode the state $\sigma(\cdot)$, the same procedure described in Section II can be applied, except that now $x(\cdot)$, $\bar{x}(\cdot)$ and $\hat{x}(\cdot)$ are replaced by, respectively, $\sigma(\cdot)$, $\varsigma(\cdot)$ and $\hat{\varsigma}(\cdot)$, and the equations which define the encoder are different (see next subsection).

C. Stabilization by encoded feedback using partial-state measurements

We first introduce the constants

$$Z := W + \mu(\gamma^*)e^{-\gamma^* t_1} 2X , \quad Y := W + Z ,$$

with W defined in Subsection IV-A, and the Lipschitz constants $F > 0$ and $H > 1$ for which

$$\begin{aligned} |f(\sigma, u) - f(\varsigma, u)|_\infty &\leq F|\sigma - \varsigma|_\infty \\ |h(\sigma) - h(\varsigma)|_\infty &\leq H|\sigma - \varsigma|_\infty \end{aligned} \quad (21)$$

for all σ, ς, u such that

$$|\sigma|_\infty, |\varsigma|_\infty \leq Y + Z, |u|_\infty \leq U .$$

Then we introduce the equations which define the center $C(t_k)$ and the range $L(t_k)$. The observer gain $\theta \geq \theta^*$ found in these equations is made explicit later on (see (23), (27)). The difference equations which define the range evolution are as follows:

$$\begin{aligned} L(t_0) &= 2X \\ L(t_1) &= 2Y \\ L(t_{k+1}) &= \Lambda L(t_k) + \Xi e^{-\gamma t_k} , \quad k \geq 1 , \end{aligned} \quad (22)$$

where, as before, $R = \Lambda/N$, $\Lambda = e^{FT}$ and

$$\Xi = 4e^{FT} |K|_\infty (\lambda_{max}(S)(\gamma + 2G))^n H \mu(\gamma) \gamma^{-1} X . \quad (23)$$

Consider also the differential equations

$$\begin{aligned} \frac{d}{dt} \varsigma(t) &= f(\varsigma(t), u(t)) , \quad t \in [t_k, t_{k+1}) , \quad k \geq 0 , \\ \varsigma(t_k) &= \hat{\varsigma}(t_k) , \end{aligned} \quad (24)$$

with initial condition $\varsigma(t_0^-) := 0$.

Remark. The difference with respect to the encoder employed before lies in the equation (22) which defines the

range. In fact, while in Section III the range was generated by an unforced difference equation, in the present case the range equation is driven by a forcing term which depends on the estimation error. This resembles an analogous expression for the range equation obtained in [5] in the case of linear discrete-time systems. \triangleleft

Reminiscent of results on semi-global stabilizability and high-gain observers, we propose an actuator which delivers the *saturated* control action $u(\cdot) = \kappa(\zeta^*(\cdot))$, with

$$\zeta^* = \begin{cases} \zeta & |\zeta|_\infty < W \\ \frac{\zeta}{|\zeta|_\infty} W & |\zeta|_\infty \geq W \end{cases} \quad (25)$$

The closed-loop system takes the form

$$\dot{x} = f(x, \kappa(\zeta^*)) \quad (26)$$

The main result of this section reads as follows:

Proposition 3: Let Assumptions 5-7 hold. For any $\rho < c + 1$ and any $\zeta < \alpha \circ \bar{\alpha}^{-1}(\rho)/(M\sqrt{n})$, if $\gamma \geq \gamma^*$ is such that

$$\Xi e^{-\gamma t_1} < \zeta \text{ and } 4\mu(\gamma)e^{-\gamma/(2MW)}X\sqrt{n} \leq \zeta \quad (27)$$

and N is such that

$$N > 2\Lambda \max\{1, Y/\zeta\}, \quad (28)$$

then the solution of the closed-loop system (26), with $\zeta^*(\cdot)$ defined through (22), (24) and (25), satisfies the following two properties:

- (i) For each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|x(t_0)|_\infty \leq L(t_0)/2 \leq \delta_\varepsilon$ implies $|x(t)|_\infty \leq \varepsilon$ for all $t \geq t_0$.
- (ii) $\lim_{t \rightarrow \infty} |x(t)|_\infty = 0$.

Proof: See [16]. \blacksquare

Remark. With no encoding involved, the conditions (cf. [18]) for the output feedback stabilization problem to be solved would reduce to

$$4\mu(\gamma)e^{-\gamma/(2MW)}X\sqrt{n} \leq \zeta$$

which is the second condition appearing in (27). We also remark that, as ρ can be chosen arbitrarily close to $c + 1$, the constant ζ depends on the size X of the set of initial conditions only, and so do the conditions (27) and (28). Hence, analogously to Proposition 2, the latter proposition gives a characterization on the observer gain and data rate required to asymptotically stabilize system (16) (when only output measurements are available) in terms of the initial uncertainty on the state. Finally, we observe that the bound (28) on N is similar to the analogous bounds in Propositions 1 and 2, and is determined from the need to have the quantization error, which depends on the range $L(\cdot)$, sufficiently small at a suitable time. With respect to those results, the main difference is that now the range is affected by the estimation error and this influences the value of the data rate employed by the encoder. \triangleleft

V. CONCLUSION

We have proposed solutions to the problem of stabilizing nonlinear systems under finite data-rate constraints. A number of scenarios have been considered, such as nonlinear systems which are integral input-to-state or asymptotically stabilizable, in the presence of full- or partial-state measurements. In the latter case, we have proposed a solution for a class of nonlinear observable systems, relying on the idea of encoding an asymptotic estimate of the state, borrowed from [5]. Several other approaches are possible. We have briefly discussed the interplay between two important design parameters, the data rate and the gain of the observer.

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